DETERMINATION OF SHAPE OF THE EARTH'S SURFACE AND MASS

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ABSTRACT

A suggestion is made for the determination of the Earth gravitational potential up to the quadropole approximation in a local determination via satellite observations. The method introduced enables one to establish also the distance of the center of the Earth to the position of the satellite observation station. This would imply that observations from several stations could be used to fix the Earth center and consequently to make observations on the shape of the Earth. A flow diagram for computations is also presented.

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1. THE POTENTIAL OF THE EARTH UP TO QUADROPOLES IN LOCAL GRAVITATIONAL DETERMINATION VIA SATELLITE OBSERVATIONS AND THE DETERMINATION OF EARTH SURFACE EQUATION AND MASS.

In the quadropole approximation the potential of the earth may be written as

$$U = \frac{GM}{r} + \frac{G\underset{\sim}{p} \cdot \underset{\sim}{r}}{r^3} + \frac{1}{2} G \sum_{i,j=1}^{3} Q_{ij} \frac{(x_i x_j)}{r^5} + \ldots , \tag{1}$$

where

$$\begin{aligned} \underset{\sim}{p} &\equiv \int \underset{\sim}{r}' \rho(x') d^3x' \\ &\equiv M\underset{\sim}{x}_c , \end{aligned} \tag{2}$$

and

$$Q_{ij} \equiv \int (3x_i' x_j' - r'^2 \delta_{ij}) \rho(x') d^3x' \quad , \tag{3}$$

with M the mass of earth, G the gravitational constant, $\underset{\sim}{r}$ the radius vector emanating from some fixed point near the center of earth, $\rho(x')$ the density of earth with x' representing the collection (x', y', z') and $\underset{\sim}{x}_c$ the vector corresponding to the center of mass of the Earth. The Q's may be interpreted in terms of the tensor moment of inertia I_{ij}.

$$I_{ij} \equiv \int x_i' x_j' \rho(x') d^3x' \quad , \tag{4}$$

as

$$I_{ij} = \frac{1}{3} Q_{ij} + \frac{1}{6} Q_{kk} \delta_{ij} \quad , \tag{5}$$

where

$$Q_{kk} \equiv Q_{11} + Q_{22} + Q_{33} .$$

Thus knowledge of the Q's enable a determination of the I's to be made. The matrix for the Q's and the I's may be written as

$$[Q_{ij}] = \begin{bmatrix} -2(I_{22}+I_{33}) & 3I_{12} & 3I_{13} \\ 3I_{12} & -2(I_{11}+I_{33}) & 3I_{23} \\ 3I_{13} & 3I_{23} & -2(I_{11}+I_{22}) \end{bmatrix} , \quad (6)$$

and

$$[I_{ij}] = \begin{bmatrix} \frac{1}{2}Q_{11}+\frac{1}{6}Q_{22}+\frac{1}{6}Q_{33} & \frac{1}{3}Q_{12} & \frac{1}{3}Q_{13} \\ \frac{1}{3}Q_{12} & \frac{1}{6}Q_{11}+\frac{1}{2}Q_{22}+\frac{1}{6}Q_{33} & \frac{1}{3}Q_{23} \\ \frac{1}{3}Q_{13} & \frac{1}{3}Q_{23} & \frac{1}{6}Q_{11}+\frac{1}{6}Q_{22}+\frac{1}{2}Q_{33} \end{bmatrix} \quad (7)$$

In what follows an indication for the determination of the properties of Earth depicted in the approximation (1) will be given insofar as it pertains to the data obtained via satellite in a region directly below the satellite. One would expect consequently that the parameters appearing in (1) will change from region to region and thus to reflect the distribution of the gravitational structure of Earth.

2. DYNAMICAL CONSIDERATIONS IN THE IDEAL CASE.

If $\underset{\sim}{V}$ is the "absolute" velocity of the satellite; $\underset{\sim}{v}$ the velocity relative to a frame rotating with Earth; $\underset{\sim}{\omega}$ the angular velocity of Earth; and $\underset{\sim}{r}$ the radius vector from the origin of the rotating axis to the satellite then

$$\underset{\sim}{V} = \underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r} \quad . \tag{8}$$

The Lagrangian may be written as

$$L = \frac{1}{2} m \underset{\sim}{V} \cdot \underset{\sim}{V} - mU \quad , \tag{9}$$

where m is the satellite mass. The introduction of (8) into (9) yields

$$L = \frac{1}{2} m (\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot (\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) - mU \quad , \tag{10}$$

so that the cannonical momenta being defined to be

$$P_i \equiv \frac{\partial L}{\partial v_i} \quad .$$

gives from (10)

$$\underset{\sim}{P} = m(\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) \quad . \tag{11}$$

The Hamiltonian is defined as

$$H = \underset{\sim}{P} \cdot \underset{\sim}{v} - L \quad , \tag{12}$$

so that from (9) and (11) in (12)

$$H = m(\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot \underset{\sim}{v} - \frac{1}{2}m(\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot (\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) + mU$$

$$= m(\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot [\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{\omega} \times \underset{\sim}{r}] + mU$$

$$= \frac{1}{2}m(\underset{\sim}{v} + \underset{\sim}{\omega} \times \underset{\sim}{r})(\underset{\sim}{v} - \underset{\sim}{\omega} \times \underset{\sim}{r}) + mU$$

$$= \frac{1}{2}m(v^2 - (\underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot (\underset{\sim}{\omega} \times \underset{\sim}{r})) + mU \quad .$$

But

$$(\underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot (\underset{\sim}{\omega} \times \underset{\sim}{r}) = \omega^2 - (\underset{\sim}{\omega} \cdot \underset{\sim}{r})^2 \quad ,$$

so that

$$\frac{H}{m} = \frac{1}{2}[v^2 + (\underset{\sim}{\omega} \cdot \underset{\sim}{r})^2 - \omega^2 r^2] + U \quad , \tag{13}$$

for the Hamiltonian divided by the mass of satellite.

Under ordinary circumstances if all perturbations are ignored then one would expect H/m to be a constant of motion. Thus a set of measurements $\underset{\sim}{v}_i$, $\underset{\sim}{r}_i$ with $i = 1,2,3,\cdots,n+1$ with n equal to or in excess of the number of parameters appearing in (1) would enable the parameters to be determined. In the former case n equations for n unknowns needs to be solved. In the latter case an appeal to least square methods may be made.

3. THE INFLUENCE OF PERTURBATIONS.

In the event that perturbations are present such as those due to the moon, sun, planets as well as atmospheric and radiation effects H/m given by (13) will be time dependent. Thus

$$\frac{d}{dt}\left(\frac{H}{m}\right) = F(t) \quad . \tag{14}$$

Hence

$$\left(\frac{H}{m}\right)_2 = \left(\frac{H}{m}\right)_1 + \int_{t_1}^{t_2} F(t)dt \quad . \tag{15}$$

Now if measurements are made in small intervals of time one would expect that the perturbing influences of distant objects such as the sun, moon, planets will be essentially constant in the time interval $t_1 < t < t_2$ and similarly for atmospheric and radiation effects. Consequently if we have in addition $t_2 < t < t_3$ we have from (15)

$$\left(\frac{H}{m}\right)_3 = \left(\frac{H}{m}\right)_2 + \int_{t_2}^{t_3} F(t)dt \quad , \tag{16}$$

so that on the basis of the additional requirement that measurements are tabulated in equal time (small) intervals

$$\int_{t_2}^{t_3} F(t)dt = \int_{t_1}^{t_2} F(t)dt \tag{17}$$

on obtains from (15) and (16) the so-called second difference equation (ignoring the m factor now)

$$H_3 - 2H_2 + H_1 = 0 \quad . \tag{18}$$

In an analogous manner one may conclude that if the F(t) is not duly constant in the time interval $t_i < t < t_{i+1}$ then an appeal to third difference equation may be made, namely

$$H_4 - 3H_3 + 3H_2 - H_1 = 0 \quad , \tag{19}$$

et cetera. The factor 1/m is to be reinstated in applications in accord with (13).

As in the ideal case we may indicate the minimum number of measurements to effect a determination of the parameters in (1). For the second difference case exemplified by (18) one has for the $i^{\underline{th}}$ measurement case to arrange for at least three measurements

$$H_{i+2} - 2H_{i+1} + H_i = 0 \quad , \tag{20}$$

with $i = 1,2,3,\cdots,n$ to yield n equations with n + 2 measurements. Thus n + 2 measurements will be necessary for the determination of n parameters appearing in (1). If the number of measurements is in excess of n + 2 least square procedures must be appealed to.

For the third difference case the analog of (20) will be

$$H_{i+3} - 3H_{i+2} + 3H_{i+1} - H_i = 0 \quad , \tag{21}$$

with $i = 1,2,3,\cdots,n$ to yield n equations with n + 3 measurements now. Thus n + 3 measurements will be necessary for the determination of n parameters appearing in (1). Again if the number of measurements is in excess of n + 3 least square procedures must be used.

Now let the assumption be made that F(t) in the neighborhood of $t = t_1$ is of the form

$$F(t) = F(t_1) + F'(t_1)(t - t_1)^2 + \frac{F''(t_1)}{2}(t - t_1)^2 + \cdot \cdot \cdot \qquad (22)$$

If in addition

$$t_2 - t_1 = t_3 - t_2$$

$$\equiv h \qquad , \qquad (23)$$

one sees that

$$\int_{t_1}^{t_2} F(t)dt = \int_{t_1}^{t_1+h} F(t)dt$$

$$= F(t_1)h + \frac{1}{2} F'(t_1)h^2 + \frac{F''(t_1)}{6}h^3 + \cdot \cdot \cdot$$

$$\int_{t_2}^{t_3} F(t)dt = \int_{t_1+h}^{t_1+2h} F(t)dt$$

$$= F(t_1)h + \frac{3}{2} F'(t_1)h^2 + \frac{7}{6} F''(t_1)h^3 \cdot \cdot \cdot$$

Hence from (15) and (16)

$$\left(\frac{1}{m}\right) [H_3 - 2H_2 + H_1] = \int_{t_2}^{t_3} F(t)dt - \int_{t_1}^{t_2} F(t)dt$$

$$= \frac{1}{2} F'(t_1)h^2 + \frac{1}{6} F''(t_1)h^3 + \cdot \cdot \cdot \qquad (24)$$

Equation (24) thus indicates the nature of the approximation associated with (20). The smallness of $F'(t_1)$ and h is essential which is essentially maintained as indicated in the discussion leading (18).

4. APPLICATION TO CASE WITH MEASUREMENTS STATED RELATIVE TO FIXED POSITION ON SURFACE OF EARTH.

Consider the axis of rotation of the Earth to be fixed in space and let this axis be called Z with unit vector $\underset{\sim}{K}$. In addition call the axis in the direction of the vernal equinox X with unit vector $\underset{\sim}{I}$. The Y-axis will be perpendicular to the X- and Z-axes with direction $\underset{\sim}{J}$ in a right handed system of coordinates. In addition consider the origin of the rotating coordinate system (local) to be situated on the axis of rotation and the z-axis to emanate outward with direction $\underset{\sim}{k}$ as unit vector. The x-axis is imagined tangent to latitude circles pointing in the east to west direction with unit vector $\underset{\sim}{i}$. The y-axis is taken perpendicular to the x- and z-axes to form a local right handed system with unit vector $\underset{\sim}{j}$ tangent to meridian circle pointing in the north to south direction. Now specify the direction of the z-axis with the spherical coordinates Θ_o, Φ_o relative to the X, Y, Z system. The relationships existing between the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$ and $\underset{\sim}{I}$, $\underset{\sim}{J}$, $\underset{\sim}{K}$ turn out to be

$$\begin{aligned} \underset{\sim}{i} &= \underset{\sim}{I} \sin \Phi_o - \underset{\sim}{J} \cos \Phi_o \quad , \\ \underset{\sim}{j} &= \underset{\sim}{I} \cos \Phi_o \cos \Theta_o + \underset{\sim}{J} \sin \Phi_o \cos \Theta_o - \underset{\sim}{K} \sin \Theta_o \quad , \\ \underset{\sim}{k} &= \underset{\sim}{I} \cos \Phi_o \sin \Theta_o + \underset{\sim}{J} \sin \Phi_o \sin \Theta_o + \underset{\sim}{K} \cos \Theta_o \quad . \end{aligned} \tag{25}$$

The inverse equations corresponding to (25) are

$$\begin{aligned} \underset{\sim}{I} &= \underset{\sim}{i} \sin \Phi_o + \underset{\sim}{j} \cos \Phi_o \cos \Theta_o + \underset{\sim}{k} \cos \Phi_o \sin \Theta_o \quad , \\ \underset{\sim}{J} &= -\underset{\sim}{i} \cos \Phi_o + \underset{\sim}{j} \sin \Phi_o \cos \Theta_o + \underset{\sim}{k} \sin \Phi_o \sin \Theta_o \quad . \end{aligned} \tag{26}$$

$$\underset{\sim}{K} = -\underset{\sim}{j} \sin \Theta_o + \underset{\sim}{k} \cos \Theta_o$$

Now since the axis of rotation is in the $\underset{\sim}{K}$ direction

$$\underset{\sim}{\omega} = \omega \underset{\sim}{K} \tag{27}$$

which in view of the last equation of (26) is

$$\underset{\sim}{\omega} = \omega (-\underset{\sim}{j} \sin \Theta_o + \underset{\sim}{k} \cos \Theta_o) \quad , \tag{28}$$

so that (13) becomes

$$\left(\frac{H}{m}\right) = \frac{1}{2} [v^2 + (y \sin \Theta_o - z \cos \Theta_o)^2 \omega^2 - \omega^2 r^2] + U$$

$$= \frac{1}{2} [v^2 + \omega^2((y \sin \Theta_o - z \cos \Theta_o)^2 - r^2)] + U \quad . \tag{29}$$

In (29) it must be remembered that

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \quad , \tag{30}$$

$$r^2 = x^2 + y^2 + z^2 \quad , \tag{31}$$

$$U = \frac{GM}{r} + \frac{GP_1 x}{r^3} + \frac{GP_2 y}{r^3} + \frac{GP_3 z}{r^3} + \frac{1}{2}\frac{GQ_{11}x^2}{r^5} + \frac{1}{2}\frac{GQ_{22}y^2}{r^5} +$$

$$+ \frac{1}{2}\frac{GQ_{33}z^2}{r^5} + \frac{GQ_{12}xy}{r^5} + \frac{GQ_{23}yz}{r^5} + \frac{GQ_{13}xz}{r^5} \quad . \tag{32}$$

It will be convenient to call

$$GM = g_o \quad ,$$

$$
\begin{aligned}
GP_1 &= a \quad , \\
GP_2 &= b \quad , \\
GP_3 &= c \quad , \\
\frac{1}{2}\, GQ_{11} &= A \quad , \\
\frac{1}{2}\, GQ_{22} &= B \quad , \\
\frac{1}{2}\, GQ_{33} &= C \quad , \\
GQ_{12} &= D \quad , \\
GQ_{23} &= E \quad , \\
GQ_{13} &= F \quad .
\end{aligned}
\tag{33}
$$

Consequently from (29), (33), (30), (31)

$$
\left(\frac{H}{m}\right) = \frac{1}{2}\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 + \omega^2[(y \sin \Theta_o - z \cos \Theta_o)^2 - r^2]\right\}
$$

$$
+ \frac{g_o}{r} + \frac{ax}{r^3} + \frac{by}{r^3} + \frac{cz}{r^3} + \frac{Ax^2}{r^5} + \frac{By^2}{r^5} + \frac{Cz^2}{r^5} + \frac{Dxy}{r^5} + \frac{Eyz}{r^5} + \frac{Fxz}{r^5} \quad , \tag{34}
$$

$$
r = (x^2 + y^2 + z^2)^{1/2} \quad . \tag{35}
$$

Also it must be remembered that the origin of the local system is presumed located on the axis of revolution and imagined to be "near" the center of mass.

Consequently, if $\underset{\sim}{R}$ is the vector from the origin on the axis to the observation point and $\underset{\sim}{\alpha}$ the satellite vector from the surface of the Earth,

$$\underset{\sim}{r} = \underset{\sim}{R} + \underset{\sim}{\alpha} \quad .$$

Hence

$$r = \sqrt{R^2 + \alpha^2 + 2\alpha \cdot \underset{\sim}{R}} \quad .$$

But

$$\underset{\sim}{R} = \underset{\sim}{R}\, R \quad ,$$

so that

$$\underset{\sim}{R} \cdot \underset{\sim}{\alpha} = R\, \underset{\sim}{\alpha} \cdot \underset{\sim}{k}$$

$$= R + \alpha_z \quad ,$$

consequently,

$$r = \sqrt{R^2 + \alpha_x^2 + \alpha_y^2 + \alpha_z^2 + 2R} \quad . \tag{36}$$

$$x = \alpha_x \quad ,$$

$$y = \alpha_y \quad , \tag{37}$$

$$z = \alpha_z \quad .$$

While (35) is useful for the case when the orbit relative to Earth is specified and information concerning the origin of the coordinate system is known (36) becomes useful in the event the magnitude of $\underset{\sim}{R}$ is not known. It will be convenient in this case to take the origin of the coordinate system to be located at the center of mass so that in (34)

$$a = b = c = 0 \quad .$$

In addition consider

$$R = R_o + \epsilon \quad , \tag{38}$$

where R_o is the mean Earth radius and ϵ is to be determined. Now since $R_o >> \epsilon$ one would expect

$$\frac{\partial r^2}{\partial R} = 2R + 2\,\frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{R}}{R} \quad , \text{ or}$$

$$\frac{\partial r}{\partial R} = \frac{1}{r}\left(R + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{R}}{R}\right) \quad , \text{ so that}$$

$$\frac{1}{r} \cong \frac{1}{r_o} - \frac{1}{r_o^3}\left(R_o + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{R_o}}{R_o}\right)\epsilon \quad , \tag{39}$$

where now

$$r_o = \sqrt{R_o^2 + \alpha^2 + 2\,R_o} \quad . \tag{40}$$

Also note from (37) and above that

$$\frac{\partial}{\partial R}[(y \sin \Theta_o - z \cos \Theta_o)^2 - r^2] = -2(y \sin \Theta_o - z \cos \Theta_o)$$

$$- 2\left(R + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{R}}{R}\right) \quad ,$$

which would indicate that

$$(y \sin \Theta_o - z \cos \Theta_o)^2 - r^2 \cong (y_o \sin \Theta_o - z_o \cos \Theta_o)^2 - r_o^2$$

$$- 2[y_o \sin \Theta_o - z_o \cos \Theta_o$$

$$+ \left(R_o + \frac{\underset{\sim}{\alpha} \cdot \underset{\sim}{R}}{R_o} \right)]\epsilon \quad . \qquad (41)$$

But also

$$\frac{dx}{dt} = \frac{d\alpha_x}{dt} \quad ,$$

$$\frac{dy}{dt} = \frac{d\alpha_y}{dt} \quad ,$$

$$\frac{dz}{dt} = \frac{d\alpha_z}{dt} \quad .$$

Since one would expect to quadropole terms to be quite small one could ignore the variation of terms of the form $\frac{\partial}{\partial R} r^{-3}$ appearing there and simply imagine the r's and x's, y's, z's there to be the r_o, x_o's, y_o's, and z_o's given by (40) and (37) with R replaced by R_o. Thus (34) becomes

$$\left(\frac{H}{m}\right) = \frac{1}{2}\{\left(\frac{d\alpha_x}{dt}\right)^2 + \left(\frac{d\alpha_y}{dt}\right)^2 + \left(\frac{d\alpha_z}{dt}\right)^2 + \omega^2[(y_o \sin \Theta_o - z_o \cos \Theta_o)^2$$

$$- r_o^2]\} + \frac{g_o}{r_o} - \{\frac{1}{r_o^3}(R_o + \alpha z) - 2\omega^2[y_o \sin \Theta_o - z_o \cos \Theta_o$$

$$+ (R_o + \alpha z)]\}\epsilon + \frac{Ax_o^2}{r_o^5} + \frac{By_o^2}{r_o^5} + \frac{Cz_o^2}{r_o^5} + \frac{Dx_o y_o}{r_o^5} + \frac{Ey_o z_o}{r_o^5} + \frac{Fx_o z_o}{r_o^5} \quad , \qquad (42)$$

where

$$\underset{\sim}{\alpha} = \underset{\sim}{i}\, \alpha x + \underset{\sim}{j}\, \alpha y + \underset{\sim}{k}\, \alpha z$$

is the position vector relative to the surface of the earth: z-axis upward, x-axis tangent to latitude circle in east to west direction and y-axis in the direction tangent to meridian circle in the north to south direction. Θ_o is the polar coordinate of measurement station

$$\begin{aligned} r_o &= \sqrt{R_o^2 + \alpha_x^2 + \alpha_y^2 + \alpha_z^2 + 2R_o\alpha_z} \\ x_o &= \alpha_x \\ y_o &= \alpha_y \\ z_o &= \alpha_z + R_o \end{aligned} \tag{43}$$

Thus (42) becomes

$$\begin{aligned} \left(\frac{H}{m}\right) = \frac{1}{2}\Big\{ & \left(\frac{d\alpha_x}{dt}\right)^2 + \left(\frac{d\alpha_y}{dt}\right)^2 + \left(\frac{d\alpha_z}{dt}\right)^2 + \omega^2[(\alpha_y \sin\Theta_o \\ & - (\alpha_z + R_o)\cos\Theta_o)^2 - r_o^2]\Big\} + \frac{g_o}{r_o} - \Big\{\frac{1}{r_o^3}(R_o + \alpha_z) \\ & - 2\omega^2[\alpha_y \sin\Theta_o - (\alpha_z + R_o)\cos\Theta_o + (R_o + \alpha_z)]\Big\}\varepsilon \\ & + \frac{A\alpha_x^2}{r_o^5} + \frac{B\alpha_y^2}{r_o^5} + \frac{C(R_o + \alpha_z^2)^2}{r_o^5} + \frac{D\alpha_x\alpha_y}{r_o^5} + \frac{E\alpha_y(R_o + \alpha_z)}{r_o^5} \\ & + \frac{F\alpha_x(R_o + \alpha_z)}{r_o^5} \end{aligned} \tag{44}$$

with r_o given by (43) and ω^2 is the angular velocity of Earth about its axis. Equation (44) is most suggestive. It indicates that we may regard g_o, ϵ, A, B, C, D, E, and F as constants to be determined. These all appear linearly in (44). In general g_o is presumed known. Nevertheless it may be of considerable interest to effect a determination of this constant. Thus, in effect, knowledge of the position vector, the position velocity relative to the surface of the Earth will enable one to estimate the mass of the earth or Newton's gravitational constant thru g_o of (33), the distance of the observation point from the center of mass of the Earth thru $\epsilon + R_o = R$ of (38), as well as the quadropole terms. Consequently the distribution of measurement stations over various portions of the surface of the Earth could be used to determine the equation of the Earth's surface since one would expect ϵ to vary from point to point.

5. PREPARATION OF POSITION MEASUREMENTS RELATIVE TO FIXED POSITIONS ON SURFACE OF EARTH CASE FOR NUMERICAL ANALYSIS.

If at least 8 + 2 measurements are presented for the $\underset{\sim}{\alpha}$ and $d\underset{\sim}{\alpha}/dt$ corresponding respectively to the position vector and velocity vector relative to a point on the surface of the Earth and the subscript k denotes the $k^{\underline{th}}$ set of measurements then if in addition the definitions

$$L_{1k} \equiv \left(\frac{1}{r_o}\right)_k \quad , \tag{45}$$

$$L_{2k} \equiv -\{\frac{1}{r_o^3} (R_o + \alpha_z) - 2\omega^2 [\alpha_y \sin \Theta_o - (\alpha_z + R_o) \cos \Theta_o + (R_o + \alpha_z)\}_k \quad , \tag{46}$$

$$L_{3k} \equiv \left(\frac{\alpha_x^2}{r_o^5}\right)_k \quad , \tag{47}$$

$$L_{4k} \equiv \left(\frac{\alpha_y^2}{r_o^5}\right)_k \quad , \tag{48}$$

$$L_{5k} \equiv \left(\frac{(R_o + \alpha_z)^2}{r_o^5}\right)_k \quad , \tag{49}$$

$$L_{6k} \equiv \left(\frac{(\alpha_x \alpha_y)}{r_o^5}\right)_k \quad , \tag{50}$$

$$L_{7k} \equiv \left(\frac{\alpha_y(R_o + \alpha_z)}{r_o^5}\right)_k \quad , \tag{51}$$

$$L_{8k} \equiv \left(\frac{\alpha_x(R_o + \alpha_z)}{r_o^5}\right)_k \quad , \tag{52}$$

$$L_{9k} \equiv -\frac{1}{2}\left\{\left(\frac{d\alpha_x}{dt}\right)^2 + \left(\frac{d\alpha_y}{dt}\right)^2 + \left(\frac{d\alpha_z}{dt}\right)^2 + \omega^2[(\alpha_y \sin \Theta_o\right.$$

$$\left. - (\alpha_z + R_o)\cos \Theta_o)^2 - r_o^2]\right\}_k \quad , \tag{53}$$

Equation (20) becomes equivalent to

$$(L_{1,k+2} - 2L_{1,k+1} + L_{1,k})g_o + (L_{2,k+2} - 2L_{2,k+1} + L_{2,k})\epsilon$$

$$+ (L_{3,k+2} - 2L_{3,k+1} + L_{3,k})A + (L_{4,k+2} - 2L_{4,k+1} + L_{4,k})B$$

$$+ (L_{5,k+2} - 2L_{5,k+1} + L_{5,k})C + (L_{6,k+2} - 2L_{6,k+1} + L_{6,k})D$$

$$+ (L_{7,k+2} - 2L_{7,k+1} + L_{7,k})E + (L_{8,k+2} - 2L_{8,k+1} + L_{8,k})F$$

$$= (L_{9,k+2} - 2L_{9,k+1} + L_{9,k}) \quad , \tag{54}$$

upon noting (44). In (54) if 8 + 2 measurements then with $k = 1, 2, \cdots 8$ one obtains 8 equations for g_o, ϵ, A, B, C, D, E, F. If max $k > 8$ a least square analysis needs to be made. In order to expedite the analysis consider the definitions

$$N_{i,k} \equiv L_{i,k+2} - 2L_{i,k+1} + L_{i,k} \tag{55}$$

where $i = 1, 2, \cdots 9$

$$\begin{aligned} A_1 &\equiv g_o \quad, \\ A_2 &\equiv \epsilon \quad, \\ A_3 &\equiv A \quad, \\ A_4 &\equiv B \quad, \\ A_5 &\equiv C \quad, \\ A_6 &\equiv D \quad, \\ A_7 &\equiv E \quad, \\ A_8 &\equiv F \quad, \end{aligned} \tag{56}$$

so that (54) reads

$$\sum_{i=1}^{8} A_i N_{i,k} = N_{1,k} \quad . \tag{58}$$

The task is to determine the A's. If max k > 8 the normal equations for the determination of the A's may be written as

$$\sum_{i=1}^{8} A_i \sum_{k=1}^{\max k} N_{i,k} N_{j,k} = \sum_{k=1}^{\max k} N_{1,k} N_{j,k} \quad , \tag{59}$$

with $j = 1, 2, \cdots 8$. If the matrix elements

$$M_{i,j} = \sum_{k=1}^{\max k} N_{i,k} N_{j,k} \quad , \tag{60}$$

is associated with the matrix M, and the row vector V has components V_j given by

$$V_j = \sum_{k=1}^{\max k} N_{9,k} N_{j,k} \quad , \tag{61}$$

the normal equations (59) may be written as

$$A^T M = V^T \quad , \tag{62}$$

where A^T is the row vector $(A_1, A_2, \cdots A_8)$. Since M is a symmetric matrix (62) may also be written as

$$MA = V \quad , \tag{63}$$

so that

$$A = (M^{-1})V \quad , \tag{64}$$

for the column vector (56). If max k = 8, then (58) has the solution

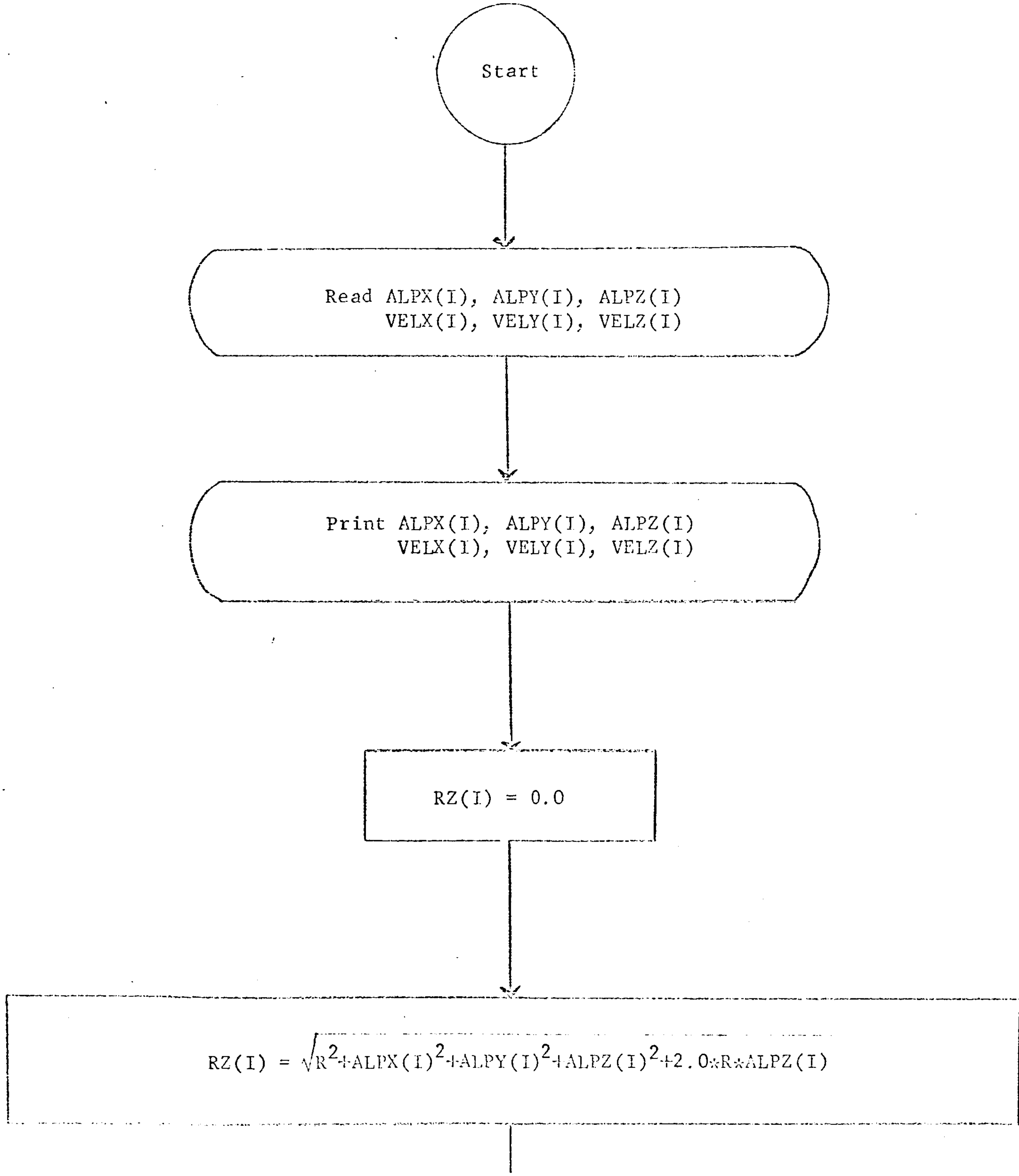
$$A = N^{-1} \begin{pmatrix} N_{1,1} \\ N_{1,2} \\ \cdot \\ \cdot \\ \cdot \\ N_{1,8} \end{pmatrix} \quad , \tag{65}$$

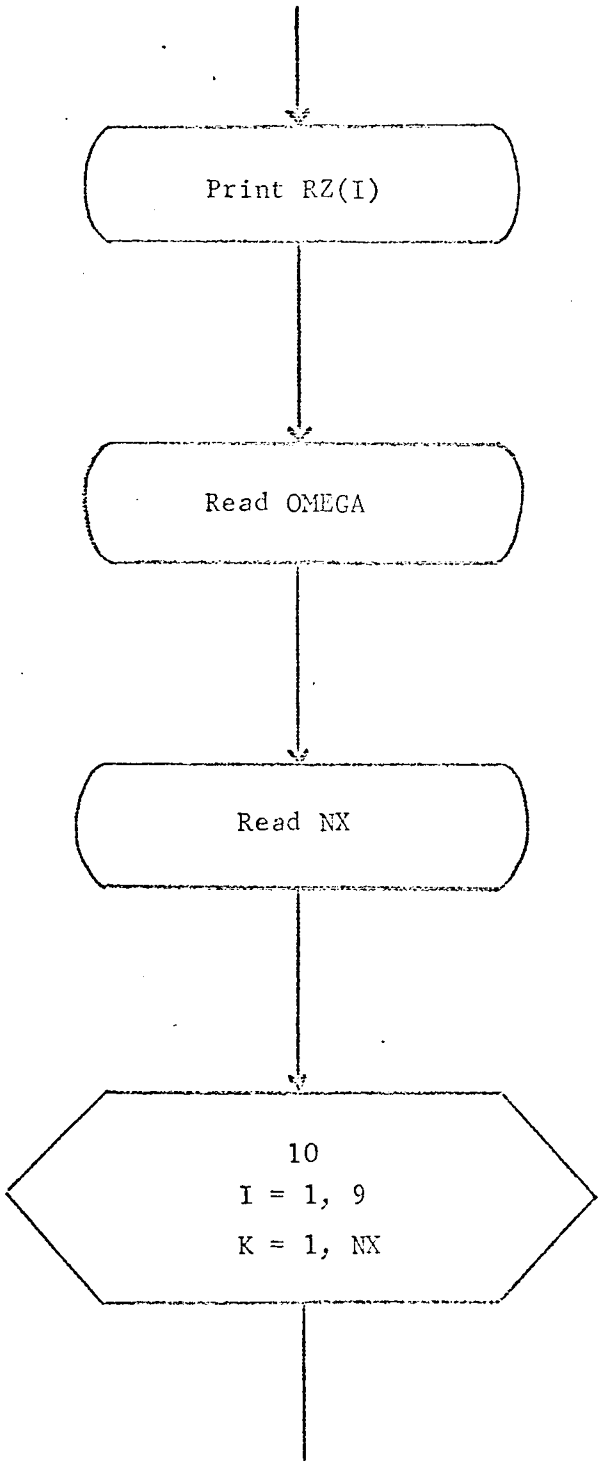
with

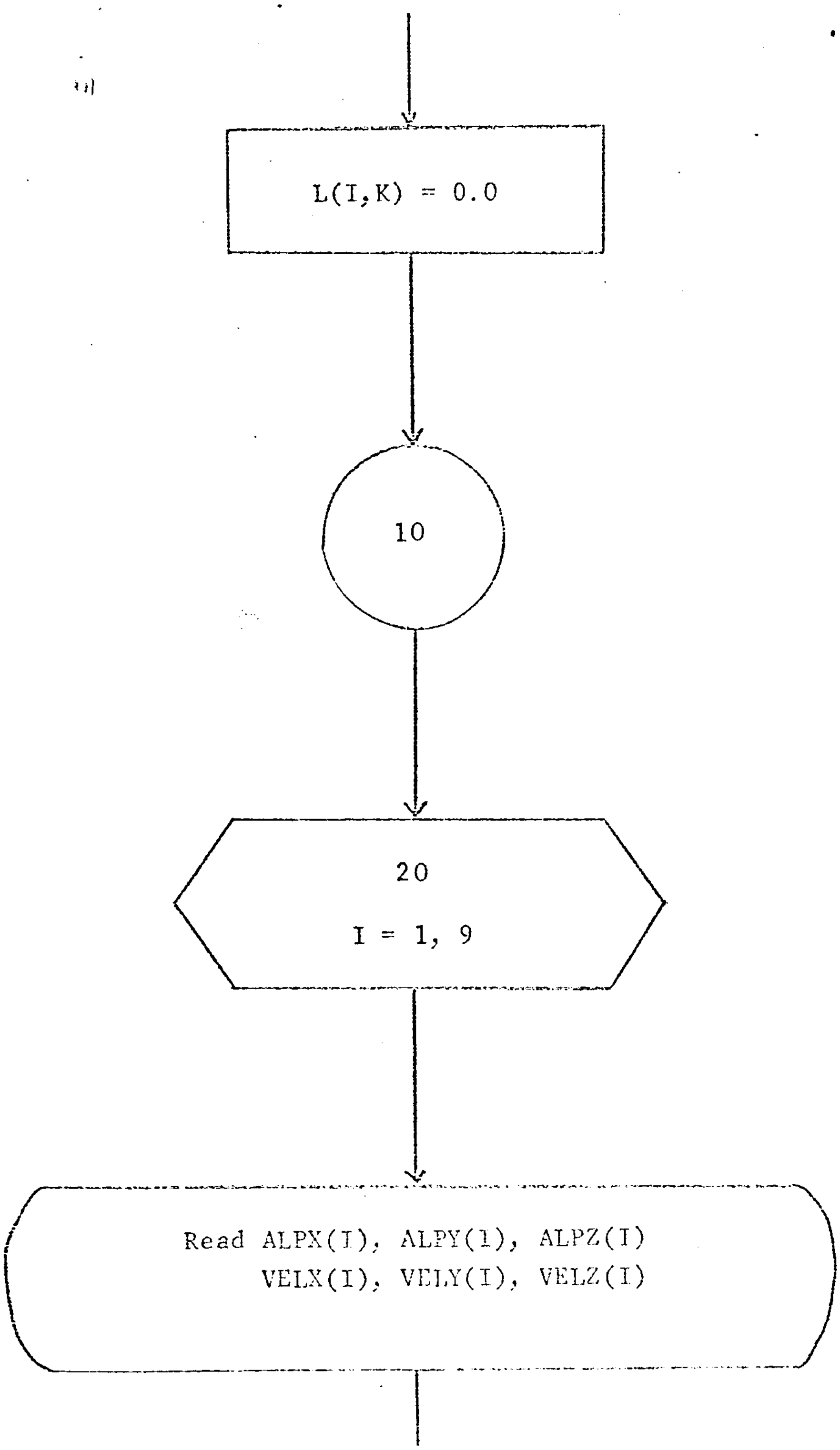
$$N \equiv \begin{pmatrix} N_{1,1} & N_{1,2} & \cdot & \cdot & \cdot & N_{1,8} \\ N_{2,1} & N_{2,2} & \cdot & \cdot & \cdot & N_{2,8} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ N_{8,1} & N_{8,2} & \cdot & \cdot & \cdot & N_{8,8} \end{pmatrix}^T , \qquad (66)$$

where T denotes transposition.

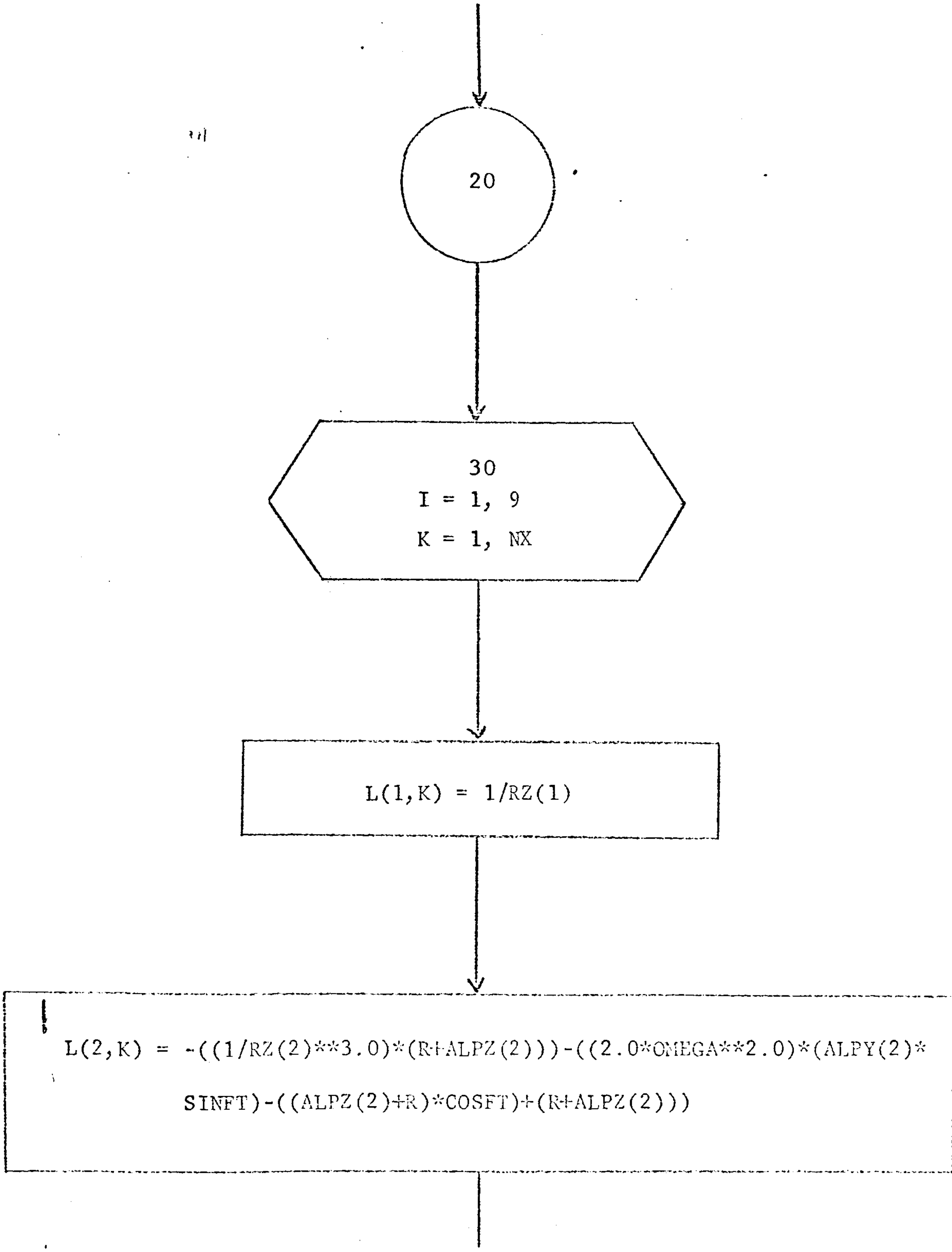
A flow diagram for the calculations follows using the definitions and references to the above equations.


Start
Read ALPX(I), ALPY(I), ALPZ(I)
VELX(I), VELY(I), VELZ(I)
Print ALPX(I), ALPY(I), ALPZ(I)
VELX(I), VELY(I), VELZ(I)
RZ(I) = 0.0
RZ(I) = √R²+ALPX(I)²+ALPY(I)²+ALPZ(I)²+2.0*R*ALPZ(I)


Print RZ(I)
Read OMEGA
Read NX
10
I = 1, 9
K = 1, NX



L(I,K) = 0.0
10
20
I = 1, 9
Read ALPX(I), ALPY(I), ALPZ(I)
VELX(I), VELY(I), VELZ(I)


20
30
I = 1, 9
K = 1, NX
L(1,K) = 1/RZ(1)
L(2,K) = -((1/RZ(2)**3.0)*(R+ALPZ(2)))-((2.0*OMEGA**2.0)*(ALPY(2)*
SINFT)-((ALPZ(2)+R)*COSFT)+(R+ALPZ(2)))

```
L(3,K) = (ALPX(3)**2.0)/RZ(3)**5.0

L(4,K) = (ALPY(4)**2.0)/RZ(4)**5.0

L(5,K) = ((R+ALPZ(5))**2.0)/RZ(5)**5.0

L(6,K) = (ALPX(6)*ALPY(6))/RZ(6)**5.0
```

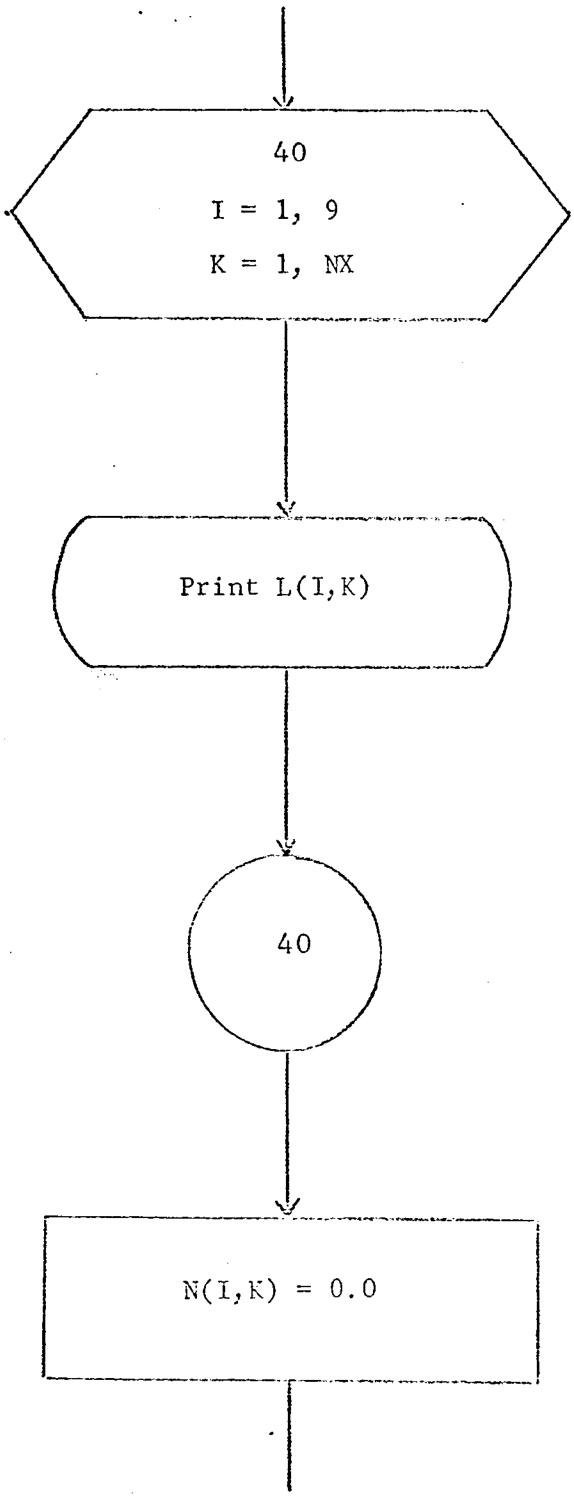
```
L(7,K) = (ALPY(7)*(R+ALPZ(7)))/RZ(7)**5.0
```

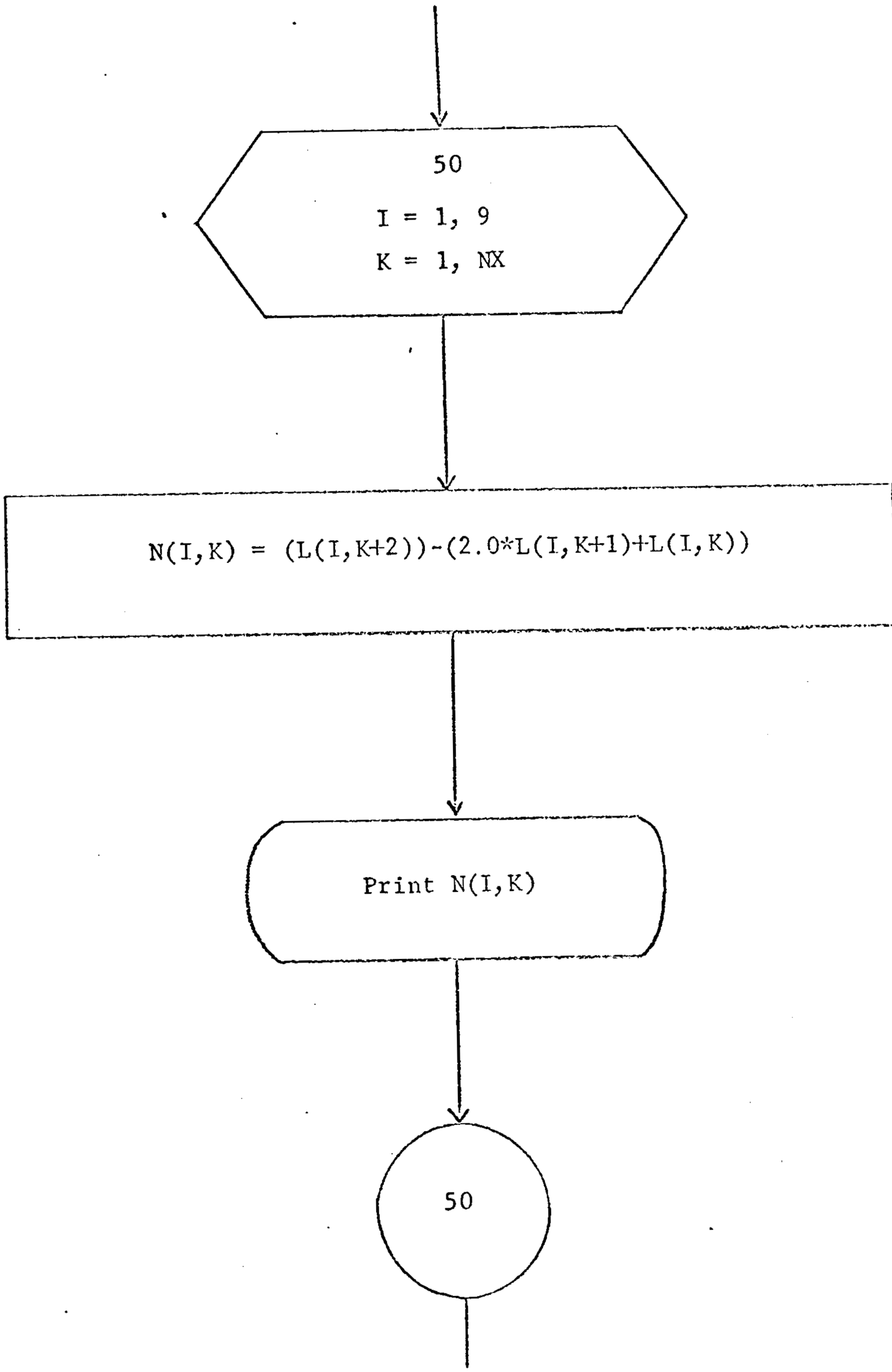
```
L(8,K) = (ALPX(8)*R+ALPZ(8)))/RZ(8)**5.0
```

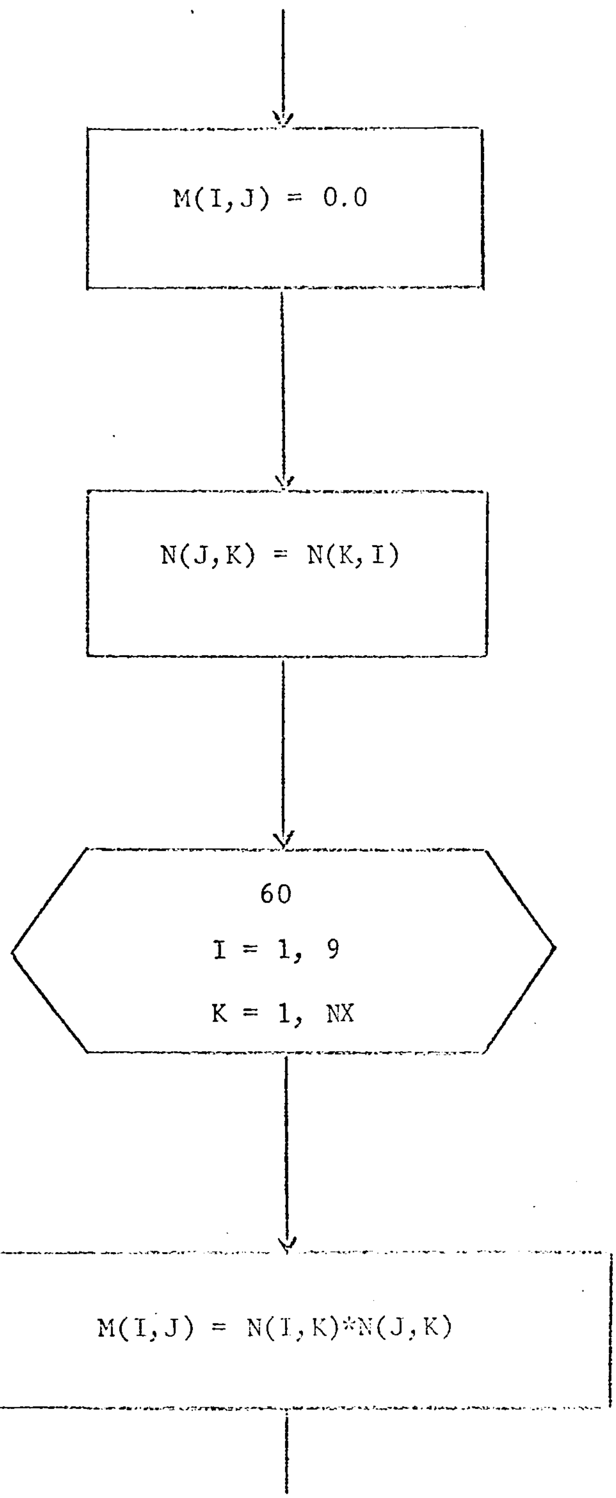
```
L(9,K) = (-1.0/2.0)*((DALPX(9)**2.0)+(DALPY(9)**2.0)+(DALPZ(9)**2.0))
         +(OMEGA**2.0)+(ALPY(9)*SINFT)-(((ALPZ(9)+R)+COSFT)**2.0)-RZ(9)**2.0)
```

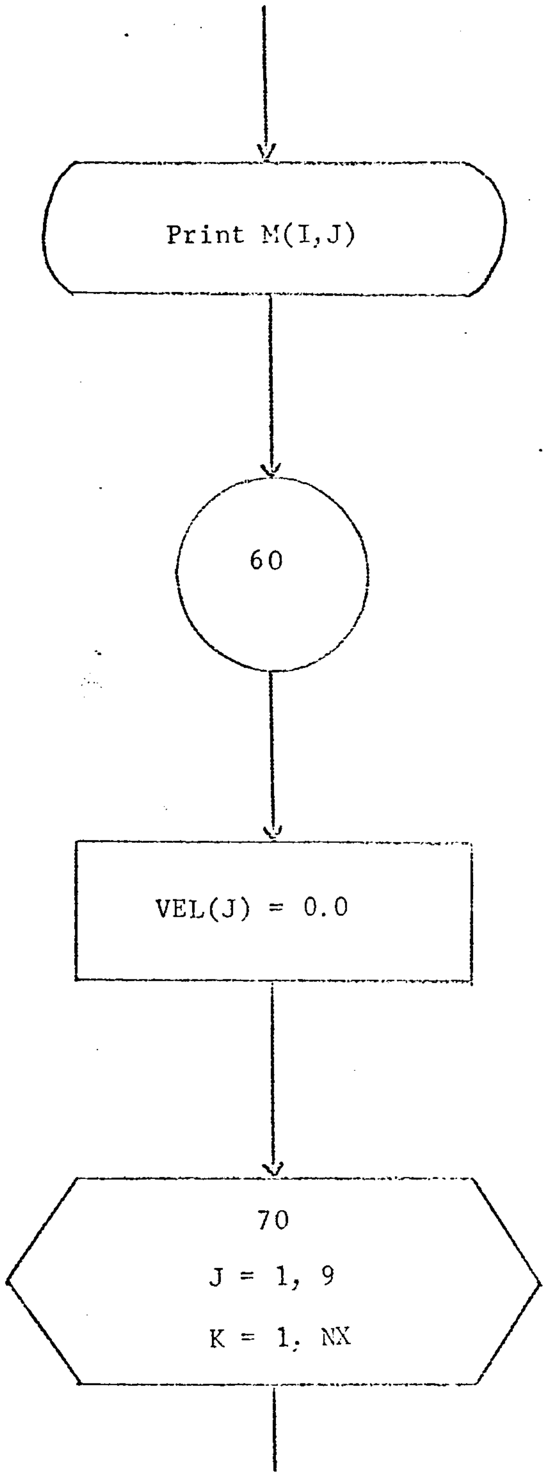
30

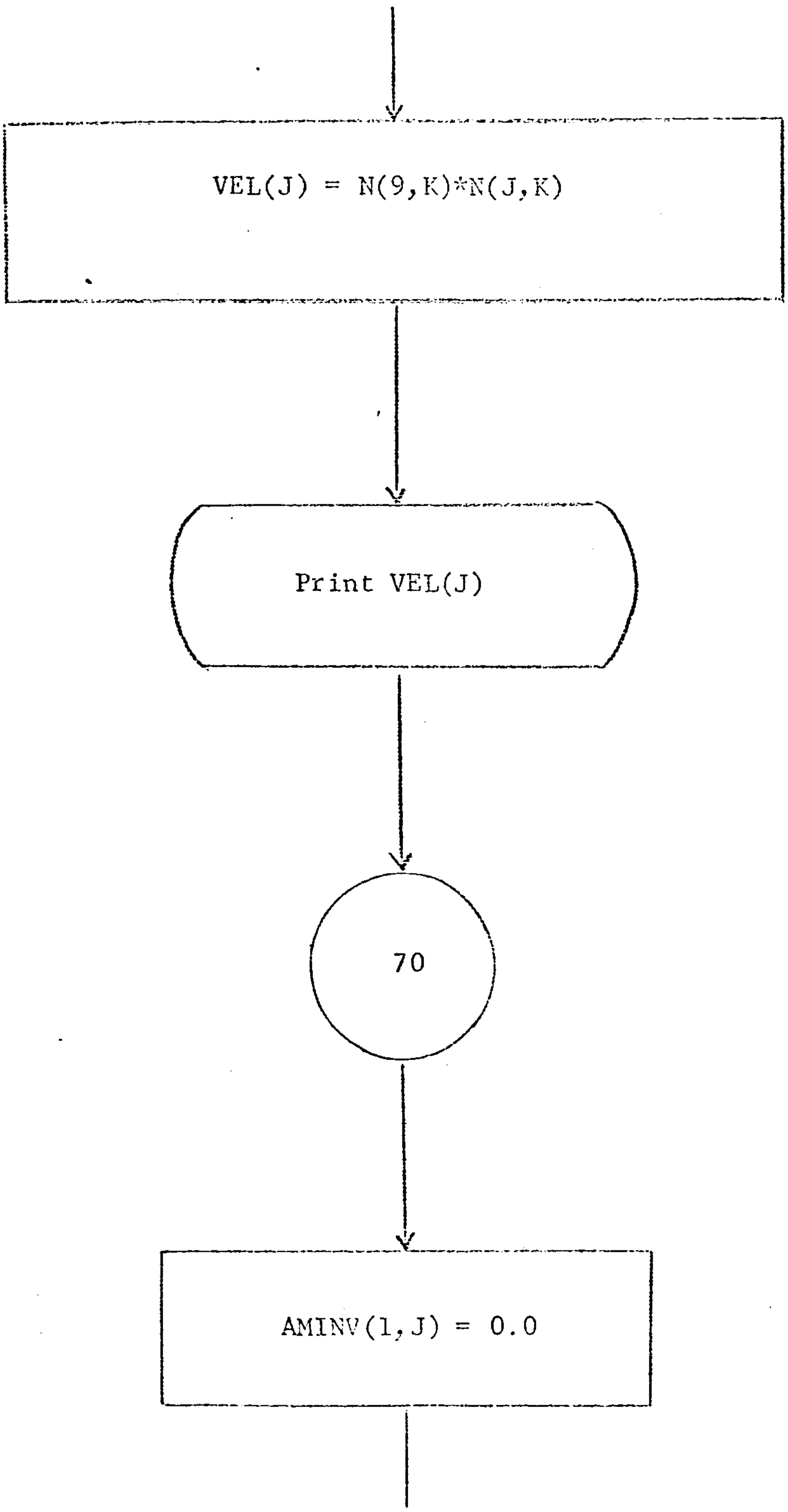
Go Back to Do 10
(until data sets
depleted)

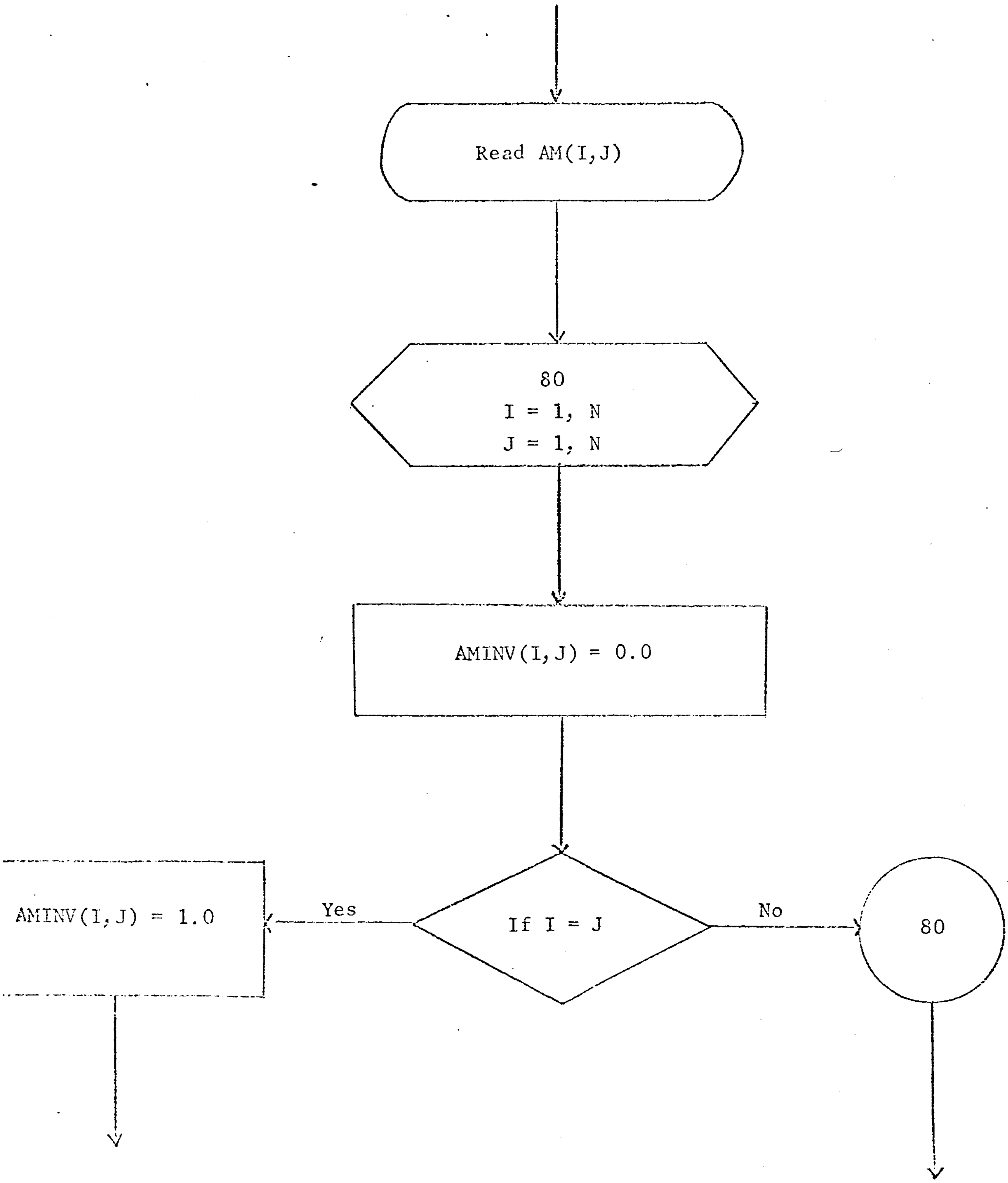

40
I = 1, 9
K = 1, NX
Print L(I,K)
40
N(I,K) = 0.0

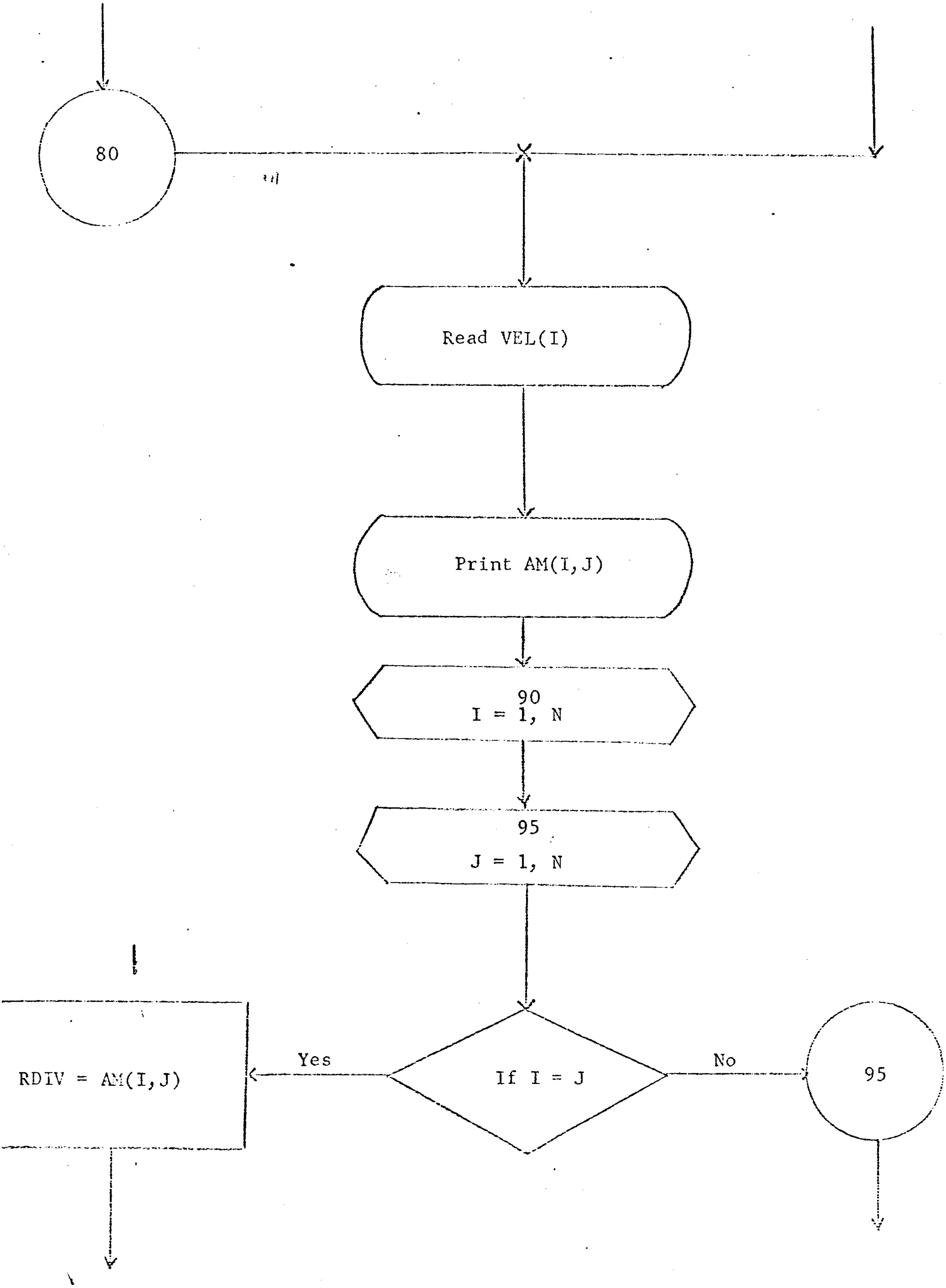

50
I = 1, 9
K = 1, NX
N(I,K) = (L(I,K+2))-(2.0*L(I,K+1)+L(I,K))
Print N(I,K)
50

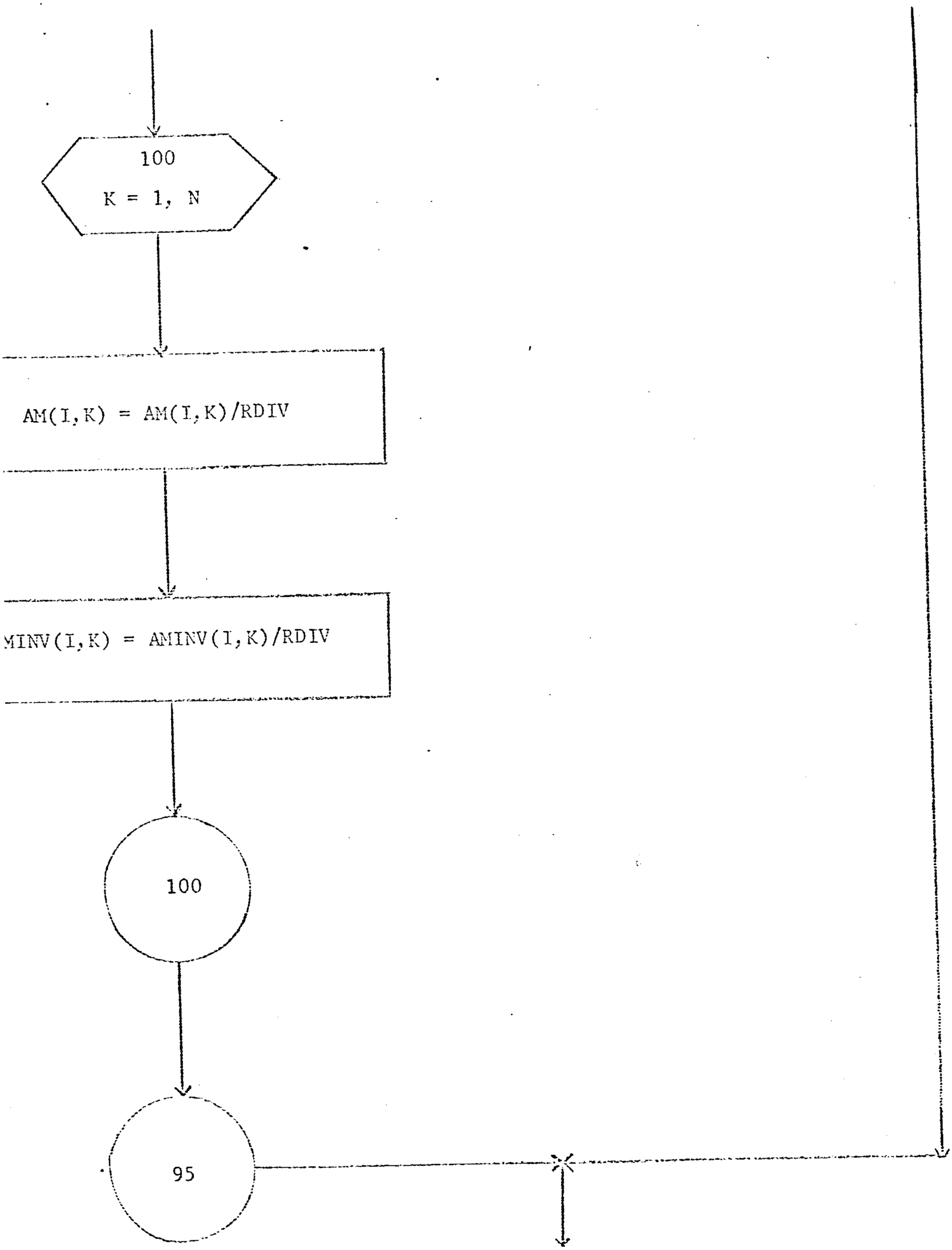

M(I,J) = 0.0
N(J,K) = N(K,I)
60
I = 1, 9
K = 1, NX
M(I,J) = N(I,K)*N(J,K)

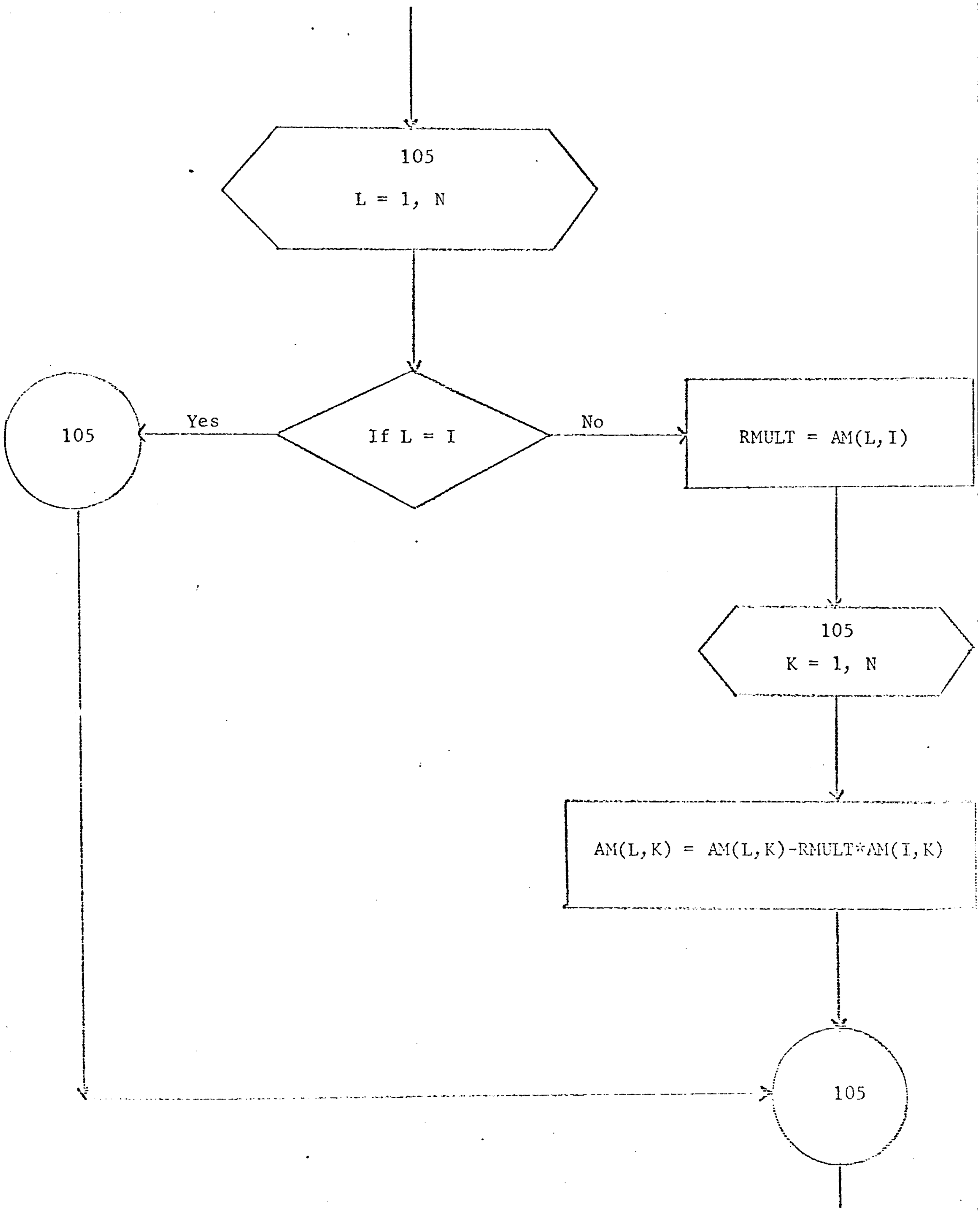

Print M(I,J)
60
VEL(J) = 0.0
70
J = 1, 9
K = 1, NX

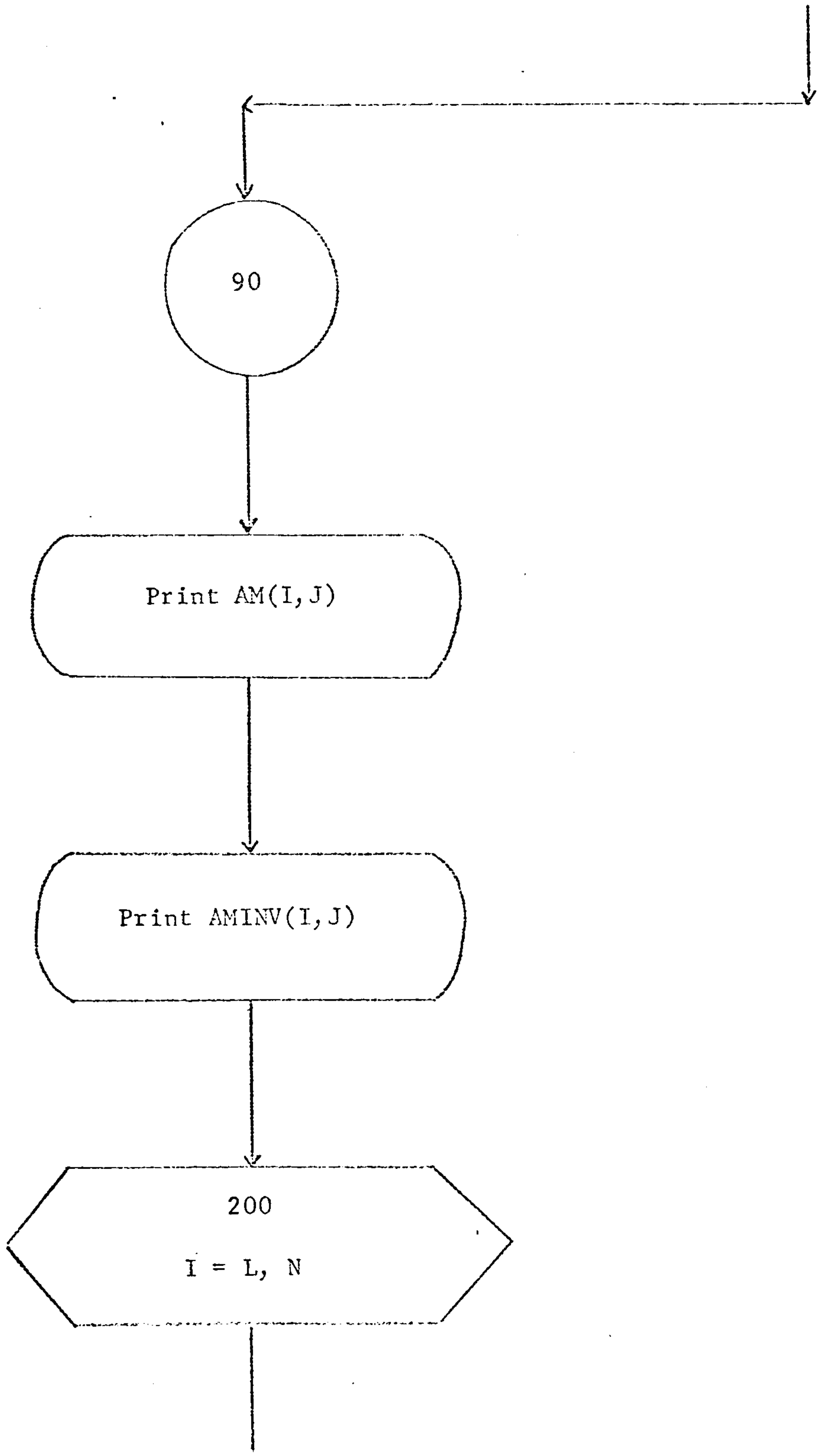

VEL(J) = N(9,K)*N(J,K)
Print VEL(J)
70
AMINV(1,J) = 0.0

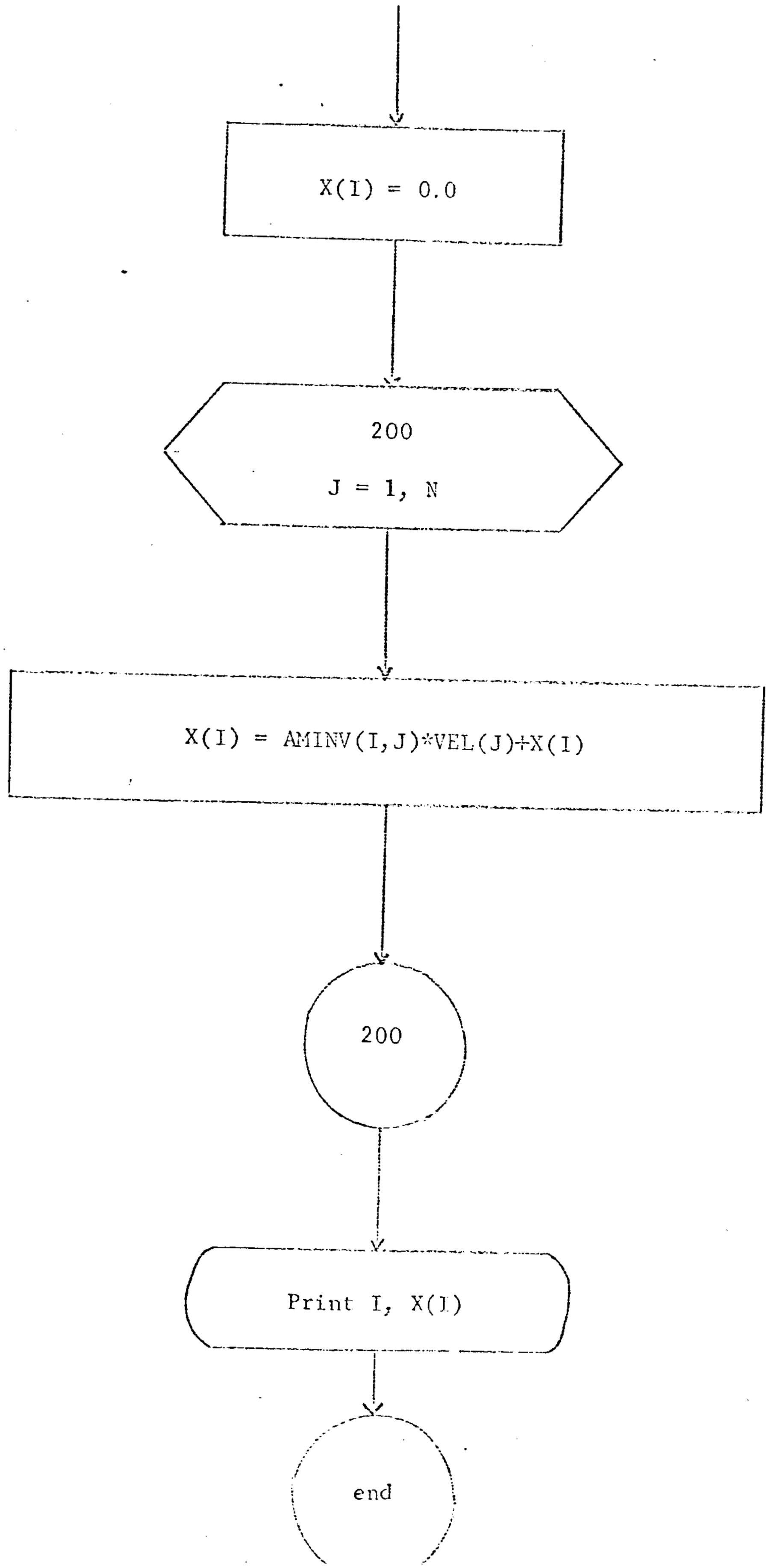

Read AM(I,J)
80
I = 1, N
J = 1, N
AMINV(I,J) = 0.0
If I = J
Yes
No
AMINV(I,J) = 1.0
80


80
Read VEL(I)
Print AM(I,J)
90
I = 1, N
95
J = 1, N
If I = J
Yes
No
RDIV = AM(I,J)
95


100
K = 1, N
AM(I,K) = AM(I,K)/RDIV
MINV(I,K) = AMINV(I,K)/RDIV
100
95


105
L = 1, N
If L = I
Yes
No
105
RMULT = AM(L,I)
105
K = 1, N
AM(L,K) = AM(L,K)-RMULT*AM(I,K)
105


90
Print AM(I,J)
Print AMINV(I,J)
200
I = L, N


X(I) = 0.0
200
J = 1, N
X(I) = AMINV(I,J)*VEL(J)+X(I)
200
Print I, X(I)
end